

On Supersymmetric Flux Solutions of M-theory

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Abstract

Motivated by the geometric structures of supersymmetric holographic RG-flows, we scan for $N = 2$ AdS_4 solutions in M-theory. One particularly well understood holographic RG flow in M-theory is dual to a mass deformation of the $\mathcal{N} = 8$ Chern-Simons theory. We utilize an Ansatz which is a natural generalization of this background in our scan. We find a single new solution with non-trivial internal flux and the topology of S^7 . Interestingly, despite our Ansatz being quite general, within our system we rule out solutions with internal flux on more general Sasaki-Einstein seven manifolds.

1 Introduction

The study of eleven dimensional supergravity provided many of the early insights into Kaluza-Klein theory and this has taken on a new significance through applications in holography [1, 2, 3]. Of particular interest are supersymmetric solutions that contain an explicit AdS_{d+1} factor with $1 \leq d \leq 6$, since each of these are expected to provide holographic duals of supersymmetric ground states in superconformal field theories.

Our understanding of holographic field theories in three dimensions was taken to a much higher level by fundamental developments on the gauge theory side [4, 5, 6] and this stimulated development and analysis of gravity duals in M theory. In particular, this provided an even greater impetus to understand better the landscape of AdS_4 solutions in eleven-dimensional supergravity. Our purpose here is to analyze a new potential class of AdS_4 solutions with fluxes and exhibit a new such supergravity solution with $\mathcal{N} = 2$ supersymmetry.

Specifically, we study solutions of eleven-dimensional supergravity in which the metric takes the form:

$$ds_{11}^2 = H^{2/3} ds_{AdS_4}^2 + H^{-1/3} ds_{M_7}^2 \quad (1)$$

and the warp factor $H = H(y^i)$ is a function of the co-ordinates, y^i , on the internal seven-manifold. We will allow for a non-vanishing membrane charge but we take the five-brane charge to be zero:

$$\int_{M_7} *F^{(4)} \sim N, \quad \int_{K_4} F^{(4)} = 0, \quad (2)$$

for all four cycles $K_4 \subset M_7$.

The prototypical class of such solutions are due to Freund and Rubin [7]

$$ds_{11}^2 = ds_{AdS_4}^2 + 6L^2 ds_{M_7}^2, \quad (3)$$

$$F^{(4)} = \frac{3}{L} \text{vol}_{AdS_4}, \quad (4)$$

where

$$Ric_4 = \frac{3}{L^2} g_4, \quad Ric_7 = \frac{3}{2L^2} g_7. \quad (5)$$

To preserve supersymmetry in a generic Freund-Rubin solution (as opposed to the special S^7 solution) there are additional geometric constraints. In particular, for $\frac{1}{4}$ -supersymmetry to be preserved, the internal manifold M_7 must be Sasaki-Einstein (SE_7) and the dual superconformal field theory in three dimensions has $\mathcal{N}=2$ supersymmetry. Seven dimensional, homogeneous, regular Sasaki-Einstein manifolds have been classified [8, 9] and the five such manifolds are usually labelled:

$$S^7, M^{3,2}, Q^{1,1,1}, N^{1,1} \text{ and } V^{5,2}. \quad (6)$$

An infinite class of non-homogeneous but explicitly known Sasaki-Einstein metrics (referred to here as the GMSW solutions) was constructed in [10] and this class of explicit SE_7 metrics was further enlarged in [11].

A natural generalization of the Freund-Rubin solutions is to allow for $F^{(4)}$ to be non-vanishing on the internal manifold, M_7 . We will consider such solutions here but our solutions will only carry dielectric five-brane charge, that is, they will not have a net five-brane charge as implied by (2). In this sense we are looking for solutions that correspond to the near horizon limit of membranes in eleven dimensions. Somewhat surprisingly, there is a distinct paucity of explicit examples of this form in the current literature. The existing solutions are related to uplifts of critical points of four dimensional, $\mathcal{N}=8$ gauged supergravity [12, 13, 14, 15]. Thanks to the proof that the $\mathcal{N}=8$ theory is a consistent truncation of eleven dimensional supergravity [16, 17, 18, 19] one can claim that any critical point of the seventy-dimensional scalar potential lifts to a solution of the $d=11$ theory although, in practice, performing this lift explicitly can be prohibitively difficult.

Only one non-trivial critical point of the $\mathcal{N}=8$ theory preserves $\frac{1}{4}$ -supersymmetry [12, 20] and this solution is characterized by the fact that it preserves $SU(3) \times U(1)$ subspace of the $\subset SO(8)$ global symmetries. The lift of the $SU(3) \times U(1)$ invariant fixed point was performed in [21] along with the entire supersymmetric domain wall [22, 23] that connects this critical point to the critical point at the origin. The lift of the $SU(3) \times U(1)$ invariant fixed point (referred to here as the CPW solution), is interesting in its own right as a solution of the

eleven-dimensional theory as it is the first explicit example of a supersymmetric AdS_4 solution with an internal flux but only electric membrane charge.

The internal manifold, M_7 , of the CPW solution has the topology of S^7 which, for our purposes, is best viewed as a fibration of $S^5 \times S^1$ over an interval I . The group of isometries of the metric is $SU(3) \times U(1)^2 \subset SO(8)$ but the four form flux is charged under one of the $U(1)$'s and thus breaks this symmetry group to $SU(3) \times U(1)$. Interestingly, one can show that, for the CPW solution, the components of $F^{(4)}$ on the internal manifold are proportional to the holomorphic two form (and its complex conjugate) on $\mathbb{CP}^2 \subset S^5$ and this is the origin of the breaking of the $U(1)$ symmetry of the Hopf fiber of S^5 . The $U(1)$ charge of the four-form flux arises in much the same manner as it does in the non-supersymmetric solutions of [24, 25] where charge of the holomorphic three-form on $\mathbb{CP}^3 \subset S^7$ is required for global regularity.

Our purpose here is to use the structure of the CPW solution as a template for more general solutions. Indeed, it was observed in [21] that while one can locally replace the $\mathbb{CP}^2 \subset S^5 \subset S^7$ by any Kähler-Einstein 4-manifold and still solve the field equations this will introduce singularities in M_7 at a particular endpoint of the interval I . Nonetheless, it is conceivable that a full analysis of the BPS equations could reveal further integration constants which allow the singularities to be avoided by judicious choices of boundary conditions of I . This is the spirit of construction of Sasaki-Einstein metrics in [10], which resolves the singularities uncovered in [26]. Our Ansatz will be designed to follow up this observation, we consider an internal seven-manifold, (8), which is $SE_5 \times U(1)$ fibered over an interval I and the three-form potential, (15), is proportional to a one-form wedged with the the holomorphic two form inherited from the Kähler-Einstein four manifold inside the SE_5 . We reduce the BPS equations for this Ansatz to a pair of O.D.E.'s

$$\begin{aligned}\frac{dS}{d\rho} &= 4 \frac{S}{\rho} - 2 H \frac{S^3}{\rho^3}, \\ \frac{dH}{d\rho} &= 2 (1 - H^2) \frac{S^4}{\rho^3(1 - S^2)}.\end{aligned}$$

In these equations, $H(\rho)$ is the warp factor of the four/seven split of the eleven-dimensional metric and is also the rotation parameter in the di-electric spinor. The function, $S(\rho)$, has a more complicated but explicit relation to the rest of the Ansatz and essentially governs how the supersymmetry mixes between the internal manifold and the radial AdS direction. Having derived the BPS equations, we also do some numerical analysis on the solution space, paying careful attention to regularity.

Interestingly, our work is orthogonal to the work [27, 28, 29, 30] where large families of $\frac{1}{4}$ -BPS, AdS_4 solutions have been constructed in IIA supergravity with non-vanishing Romans mass [31]. Solutions with non-vanishing Romans mass cannot be lifted to eleven-dimensional supergravity and, conversely, our Ansatz cannot be reduced to supersymmetric solutions of IIA supergravity because the residual $U(1)$ symmetry in our Ansatz is the \mathcal{R} -symmetry and so compactification along this $U(1)$ will break the supersymmetry.

This paper is organized as follows: In section 2 we outline our Ansatz and the methods we have used to reduce the BPS conditions to a pair of ODE's. In section 3 we review the

known analytic solutions to this Ansatz. In section 4 we numerically solve the pair of ODE's. We reproduce the known analytic solution and produce a new solution with the topology of S^7 .

Note added: While preparing this paper for submission we became aware of the work [32] which may have some overlap with our paper.

2 The Supersymmetry Conditions

2.1 The Background

While our solution space will ultimately be determined by just two functions, $H(\rho)$ and $S(\rho)$ we first outline our complete Ansatz. The metric is:

$$ds_{11}^2 = H^{2/3} ds_{AdS_4}^2 + 6L^2 H^{-1/3} ds_7^2, \quad (7)$$

where

$$ds_7^2 = X_2^2 d\rho^2 + \rho^2 ds_{B_2}^2 + (X_4(d\psi + A) + X_5 d\phi)^2 + X_6^2 d\phi^2, \quad (8)$$

and all functions depend only on ρ , which parametrizes the interval, I . The end-points of I will be determined by regularity. The base, B_2 , is taken to be a Kahler-Einstein manifold, normalized by $R_{\alpha\beta} = 6g_{\alpha\beta}$. This metric (8) expresses M_7 as a fibration of $SE_5 \times U(1)_\phi$ over I .

To analyze the spinors, we choose the frames:

$$\begin{aligned} e^{1,2,3} &= H^{1/3} e^{r/L} dx^{0,1,2}, \\ e^4 &= \sqrt{1 - S^2} H^{1/3} dr + \sqrt{6} L H^{-1/6} S X_2 d\rho, \\ e^5 &= S H^{1/3} dr - \sqrt{6} L H^{-1/6} \sqrt{1 - S^2} X_2 d\rho, \\ e^{6,7,8,9} &= \sqrt{6} L H^{-1/6} \rho f^{1,2,3,4}, \\ e^{10} &= \sqrt{6} L H^{-1/6} (X_4(d\psi + A) + X_5 d\phi), \\ e^{11} &= \sqrt{6} L H^{-1/6} X_6 d\phi. \end{aligned} \quad (9)$$

The frames, f^i , are chosen so that the canonical holomorphic frames on B_2 are:

$$\mathfrak{f}^1 = f^1 - i f^4, \quad \mathfrak{f}^2 = f^2 - i f^3. \quad (10)$$

Then the complex-structure and (local) holomorphic $(2,0)$ -form on B_2 are:

$$j \equiv \frac{i}{2} (\mathfrak{f}^1 \wedge \bar{\mathfrak{f}}^1 + \mathfrak{f}^2 \wedge \bar{\mathfrak{f}}^2), \quad \omega \equiv i \mathfrak{f}^1 \wedge \mathfrak{f}^2, \quad (11)$$

and we have:

$$dA = 2j, \quad d\omega = 3iA \wedge \omega, \quad \omega \wedge \omega^* = 2j \wedge j. \quad (12)$$

We can also define the fundamental forms of a SE_5 structure:

$$\eta = (d\psi + A), \quad J = j, \quad \Omega = e^{3i\psi}\omega, \quad (13)$$

which satisfy

$$\Omega \wedge \Omega^* = 2J \wedge J, \quad d\eta = 2J, \quad dJ = 0, \quad d\Omega = 3i\eta \wedge \Omega. \quad (14)$$

Following [21], we assume that the internal components of the three-form potential are proportional to the holomorphic $(2, 0)$ form, Ω :

$$C^{(3)} = p_0 e^1 \wedge e^2 \wedge e^3 + \text{Re}[(a_1 d\rho + a_2 (d\psi + A) + a_3 d\phi) \wedge \Omega] \quad (15)$$

where p_0 is a real function of ρ and the a_j are complex functions of ρ . Note that there AdS -invariance implies that there can be no internal parts proportional to dr .

2.2 The Supersymmetry Projectors

Using techniques similar to [33], we require that $\frac{1}{4}$ -supersymmetry is preserved. This requires that the eleven-dimensional spinor, ϵ , satisfies three projection conditions. Define

$$\Pi_0 = \frac{1}{2}(\mathbf{1} + \Gamma^{123}), \quad \Pi_1 = \frac{1}{2}(\mathbf{1} + \Gamma^{78}\Gamma^{69}), \quad \Pi_2 = \frac{1}{2}(\mathbf{1} - \Gamma^{78}\Gamma^{510}). \quad (16)$$

and the di-electric projector:

$$\tilde{\Pi}_0(\xi, \beta) = \frac{1}{2}(\mathbf{1} + \cos \beta \Gamma^{123} - \sin \beta \Gamma^{123}(\cos \xi \Gamma^{7910} + \sin \xi \Gamma^{8910})), \quad (17)$$

The Killing spinor, ϵ , that we seek satisfies:

$$\tilde{\Pi}_0(3\psi, \beta)\epsilon = \Pi_1\epsilon = \Pi_2\epsilon = 0, \quad (18)$$

If one introduces the rotations

$$\mathcal{R}(x) = \cos \frac{x}{2} - \sin \frac{x}{2} \Gamma^{78}, \quad (19)$$

and

$$\mathcal{O}(x) = \cos \frac{x}{2} + \sin \frac{x}{2} \Gamma^{7910}. \quad (20)$$

then the Killing spinor may be written as

$$\epsilon = H^{1/6} e^{r/(2L)} \mathcal{R}(3\psi) \mathcal{O}(\beta) \mathcal{R}(-2\phi) \epsilon_0. \quad (21)$$

where ϵ_0 is a *constant* spinor satisfying the simpler projections:

$$\Pi_0\epsilon_0 = \Pi_1\epsilon_0 = \Pi_2\epsilon_0 = 0, \quad (22)$$

The rotations (19) and (20) commute with Π_1 and Π_2 , but rotate with Π_0 in $\tilde{\Pi}_0$, and $\mathcal{O}(\beta)$, in particular, represents a partial polarization of the $M2$ -branes into dielectric $M5$ -branes.

3 The Analytic Solution Space

3.1 The Supersymmetry Conditions

It is now relatively easy to solve the supersymmetry conditions based upon our Ansatz for the background (9), (15) and the supersymmetries (21).

We find that everything can be parametrized the two functions, $H(\rho)$ and $S(\rho)$, that were used in the definition of our frames (9). The remaining metric functions may then be written as

$$\begin{aligned} U(\rho)^{-1/2} &\equiv X_2 = -\sqrt{\frac{2}{3}} \sqrt{H} \frac{S}{\rho \sqrt{1-S^2}}, & X_4 &= \sqrt{\frac{3}{2}} \frac{1}{\sqrt{H}} \frac{\rho^2}{S}, \\ X_5 &= -\frac{1}{\sqrt{6}} \frac{1}{\sqrt{H}} S, & X_6 &= \frac{1}{\sqrt{6}} \sqrt{H} \sqrt{1-S^2}. \end{aligned} \quad (23)$$

We have defined $U(\rho)$ in analogy with the same function in [10]. The di-electric angle is given by:

$$\cos \beta = \frac{1}{H}, \quad \sin \beta = \left(1 - \frac{1}{H^2}\right)^{\frac{1}{2}}. \quad (24)$$

The three-form potential can be simplified to the form:

$$C^{(3)} = p_0 e^1 \wedge e^2 \wedge e^3 + 6L^2 \rho^2 H^{-1/3} \operatorname{Re} \left[-i \left(p_1 (e^4 + ie^{11}) + p_2 (e^5 + ie^{10}) \right) \wedge \Omega \right], \quad (25)$$

where we must have $S p_2 = -(\sqrt{1-S^2}) p_1$ in order to cancel the dr components. Indeed we find:

$$p_0 = -\frac{1}{2} \cos \beta, \quad p_1 = \frac{S}{\sqrt{1-S^2}} \frac{1}{2} \tan \frac{\beta}{2}, \quad p_2 = -\frac{1}{2} \tan \frac{\beta}{2}. \quad (26)$$

Finally, the remaining supersymmetry conditions reduce to the first order system on H and S :

$$\begin{aligned} \frac{dS}{d\rho} &= 4 \frac{S}{\rho} - 2 H \frac{S^3}{\rho^3}, \\ \frac{dH}{d\rho} &= 2 (1 - H^2) \frac{S^4}{\rho^3 (1 - S^2)}. \end{aligned} \quad (27)$$

These equations thus completely determine our solutions.

3.2 The GMSW Sasaki-Einstein Solutions

The solutions of [10] are obtained from a particularly simple set of solutions to (27)

$$H = 1, \quad S(\rho) = \sqrt{\frac{3}{2}} \frac{\rho^4}{\sqrt{c + \rho^6}}. \quad (28)$$

This implies that $\beta = 0$ and hence there are no fluxes. The metric function, $U(\rho)$, also becomes:

$$U(\rho) = \left(1 - \frac{3}{2} \rho^2 + \frac{c}{\rho^6} \right), \quad (29)$$

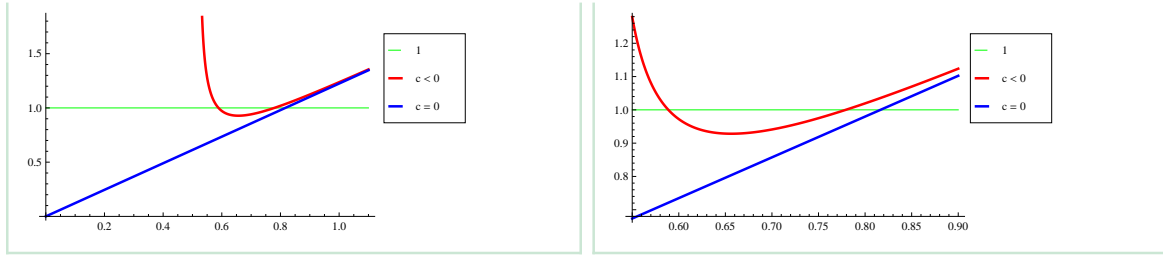


Figure 1: The GMSW solution: $S(\rho)$ for $c = 0$ and $c_0 < c < 0$.

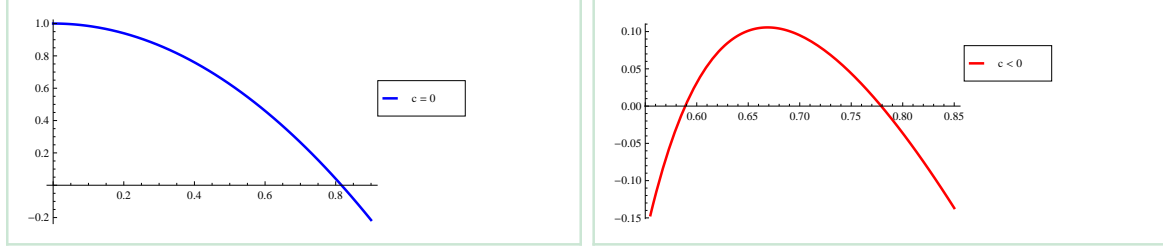


Figure 2: The GMSW solution: $U(\rho)$ for $c = 0$ and for $c_0 < c < 0$.

where c is the integration constant, proportional to κ in [10]. When $c = 0$, $KE_4 = \mathbb{CP}^2$ and we obtain $M_7 = S^7$ and $0 < \rho < \sqrt{2/3}$. At $\rho = 0$ the space is obviously smooth while at the rightmost endpoint, there is a bolt.

The GMSW solutions have $c < 0$ in the range where $U(\rho)$ is finite and positive on some interval $0 < \rho_i < \rho < \rho_{i+1}$. On that interval we must have $0 < S(\rho) \leq 1$, where the equality holds only at the zeros, ρ_i . The plots illustrating this are in Fig. 1 and 2. Regularity of these solutions at the endpoints can be checked analytically. These solutions can be constructed upon any KE_4 and one finds an infinite discrete family parameterized by c .

3.3 The CPW solution

The CPW solution is obtained by setting

$$H(\rho) = 3 - 2\rho^2, \quad S = \frac{\rho}{\sqrt{2 - \rho^2}}, \quad 0 \leq \rho \leq 1. \quad (30)$$

The solution is, of course, smooth but it is useful to examine the details in our parametrization. At the leftmost endpoint of I , we have

$$ds_7^2 = d\rho^2 + \rho^2 \left[ds_{B_2}^2 + \left(d\psi - \frac{1}{6} d\phi + A \right)^2 \right] + \left(\frac{1}{2} - \frac{21}{36} \rho^2 \right) d\phi^2 + \mathcal{O}(\rho^4). \quad (31)$$

First note that the S^1 defined by the ϕ -circle remains large as $\rho \rightarrow 0$. If one sets $d\phi = 0$ then the remaining metric is simply a cone over the $U(1)$ Hopf fibration over \mathbb{CP}^2 . If the angle, ψ , has period 2π then this is a cone over S^5 and is thus precisely the flat metric on \mathbb{R}^6 ; exactly as one should expect if one describes a smooth S^7 as $S^5 \times S^1$ fibered over the interval, I .

At the rightmost endpoint of I , we set $(1 - \rho) = \frac{1}{4}x^2$ and find

$$ds_7^2 \sim \frac{1}{6} \left(dx^2 + x^2 (d\phi - \frac{1}{2}\chi)^2 \right) + ds_{B_2}^2 + \frac{1}{6} \left(1 - \frac{5}{4}x^2 \right) \chi^2, \quad (32)$$

where $\chi \equiv d\phi + 3(d\psi + A)$. The \mathbb{CP}^2 and the fiber defined by the differential, χ , now remain large as $x \rightarrow 0$. Dropping the $ds_{B_2}^2$ and χ terms leaves $dx^2 + x^2 d\phi^2$, which defines a smooth \mathbb{R}^2 if and only if ϕ has period 2π . Again, this is precisely what one expects for a smooth S^7 . Note that the regularity of the metric determines the periods of the angular coordinates ψ and ϕ as it did in [10].

4 The Numerical Solution Space

The two distinct classes of known solutions to (27) are classified by whether $U(\rho)$ has a zero at both endpoints of I or just the rightmost endpoint. The SE_7 solutions are of the first kind while the round and squashed (CPW) S^7 solutions are of the second kind. We now investigate solutions of both kind, starting by characterizing the zeros, ρ_i , of $U(\rho)$.

4.1 Expanding around a zero of $U(\rho)$

We can tabulate the zero's of $U(\rho)$ as follows. It is clear from (27) that at such a regular zero, ρ_i , we must have

$$S(\rho_i) = H(\rho_i) = 1. \quad (33)$$

Using series expansions

$$S(\rho) = 1 + \sum_{n \geq 1} s_n (\rho - \rho_i)^n, \quad H(\rho) = 1 + \sum_{n \geq 1} h_n (\rho - \rho_i)^n, \quad (34)$$

we find that the constant terms in the expansion of (27) yield:

$$s_1 - \frac{4}{\rho_i} + \frac{2}{\rho_i^3} = 0, \quad h_1 \left(1 - \frac{2}{s_1 \rho_i^3} \right) = 0, \quad (35)$$

while the higher terms have the structure

$$s_n + \sigma_n(s_k, h_k) = 0, \quad h_n \omega_n(s_k, h_k) + \mu_n(s_k, h_k) = 0, \quad k = 1, \dots, n-1, \quad (36)$$

where

$$\mu_n(s_k, h_k) = 0 \quad \text{if} \quad h_1 = h_2 = \dots = h_{n-1} = 0. \quad (37)$$

Suppose first that $h_1 \neq 0$. Then the consistency of recurrence arises at the first step and (35) implies

$$\rho_0 = 1, \quad s_1 = 2. \quad (38)$$

The higher order terms yield a unique solution in terms of a single parameter, h_1 .

However, if we have $h_1 = \dots = h_i = 0$ and $h_{i+1} \neq 0$, the first i -steps of the recurrence are trivially solved for s_1, \dots, s_i in terms of ρ_i . Now, the consistency condition arises at order $i+1$

where we must solve $\omega_i = 0$, which is an equation for the i -th zero ρ_i . This determines the discrete set of zeros (39). The higher orders are then solved without any further conditions and yield a family of solutions parametrized by h_{i+1} .

So we obtain the result that the analytic zeros of $U(\rho)$ are given by:

$$\rho_0 = 1, \quad \rho_1 = \frac{\sqrt{3}}{2}, \quad \dots \quad \rho_j = \sqrt{\frac{j+2}{2(j+1)}}, \quad \dots \quad (39)$$

and hence:

$$\rho_j s_1 \equiv \rho_j S'(\rho_j) = \frac{4}{j+2}. \quad (40)$$

Also note that

$$H'(\rho_i) = \dots = H^{(i)}(\rho_i) = 0, \quad H^{(i+1)}(\rho_i) \neq 0. \quad (41)$$

The fact that $S'(\rho_i) > 0$ imposes a strong constraint on solutions because it directly implies that if $H(\rho) \neq 1$ then at most one such zero can be present in any given solution.

It is useful to analyze the metric around the points, ρ_j . We define $(\tilde{\phi}, \tilde{\psi}) = (\phi, \phi - 3\rho_j^2\psi)$ and the one-form $\chi = d\tilde{\psi} + A$ then find

$$\begin{aligned} ds_7^2 &= \frac{1}{3(\rho_j^2 s_1)} \frac{d\rho^2}{\rho_j - \rho} + \frac{s_1}{3} (\rho_j - \rho) \left(d\tilde{\phi} + \left(\frac{1}{\rho_j s_1} - 1 \right) \chi \right)^2 \\ &\quad + \rho_j^2 ds_{B_4}^2 + \frac{1}{6} \left(1 + (\rho_j - \rho) \left(h_1 - \frac{2}{s_1 \rho_j^2} \right) \right) \chi^2 + \mathcal{O}((\rho_j - \rho)^2). \end{aligned} \quad (42)$$

After the change of co-ordinates $\rho_j - \rho = \frac{1}{4}x^2$ the $(x, \tilde{\phi})$ part of the metric is

$$ds_{(x, \tilde{\phi})}^2 \sim dx^2 + \frac{\rho_j^2 s_1^2}{4} x^2 d\tilde{\phi}^2 = dx^2 + x^2 \left(\frac{2}{(j+2)} d\tilde{\phi} \right)^2, \quad (43)$$

where we have used (40). Regularity at $x = 0$ means that $\frac{2}{(j+2)}\tilde{\phi}$ must have period 2π and since $\tilde{\phi} = \phi$, we have

$$\phi \equiv \phi + (j+2)\pi. \quad (44)$$

From (21) we see that, for the supersymmetry to be well-defined, ϕ can be given a periodicity of

$$\phi \equiv \phi + n\pi, \quad n \in \mathbb{Z}, \quad (45)$$

and we also note that the flux defined by (25) and (13) is independent of ϕ . Therefore the identification (44) is compatible with our background and its supersymmetry.

4.2 Expanding $\rho = 0$

Unless $U(\rho) \rightarrow 0$ at some leftmost endpoint of I , the interval I must terminate at $\rho = 0$ because the \mathbb{CP}^2 pinches off at this point. Analytic solutions at $\rho = 0$ fall into two distinct families. The first family depends on two parameters (s, h)

$$\begin{aligned} S(\rho) &= s\rho^4 - \frac{1}{3}hs^3\rho^{10} + \frac{1}{6}h^2s^5\rho^{16} + \frac{5}{54}h^3s^7\rho^{22} - \frac{1}{70}(1-h^2)s^7\rho^{24} + \dots, \\ H(\rho) &= h + \frac{1}{7}(1-h^2)s^4\rho^{14} - \frac{2}{15}h(1-h^2)s^6\rho^{20} - \frac{1}{11}(1-h^2)s^6\rho^{22} + \dots \end{aligned} \quad (46)$$

The second family has just one parameter (s) and a “linear behaviour” at $\rho \rightarrow 0$,

$$\begin{aligned} S(\rho) &= s\rho + \left(\frac{9s^3}{16} - \frac{s^7}{4}\right)\rho^3 + \left(\frac{1359s^5}{2560} - \frac{43s^9}{64} + \frac{31s^{13}}{160}\right)\rho^5 + \dots, \\ H(\rho) &= \frac{3}{2s^2} - \left(\frac{9}{4} - s^4\right)\rho^2 - \left(\frac{9s^2}{32} - \frac{5s^6}{4} + \frac{s^{10}}{2}\right)\rho^4 + \dots \end{aligned} \quad (47)$$

For this family, the metric around $\rho = 0$ is

$$ds_7^2 = d\rho^2 + \rho^2 \left(ds_{B_2}^2 + \left(d\psi - \frac{s^2}{3} d\phi + A_2 \right)^2 \right) + \left(\frac{1}{4s^2} + \frac{1}{72} (12s^4 - 45)\rho^2 \right) d\phi^2 + \mathcal{O}(\rho^4), \quad (48)$$

so we see that for regularity the range of ψ is given by

$$0 \leq \psi \leq 2\pi. \quad (49)$$

This is consistent with the flux (15) and the supersymmetry (21).

From (23) we see that for this second, “linear” class, $U(\rho)$ goes to a constant as $\rho \rightarrow 0$ while the asymptotics (46) implies that $U(\rho)$ diverges as $\rho \rightarrow 0$. Thus regular solutions must have the linear asymptotics of (47).

4.3 Solutions with $U(\rho_0) = 0$

The CPW-solution (30) is plotted in Fig 3. Expanding the rightmost endpoint of I it has $U(\rho_0) = 0$ with $h_1 = -4$. Around the leftmost endpoint it belongs to the second family and has $U(0) = 1$ and $s = \frac{1}{\sqrt{2}}$.

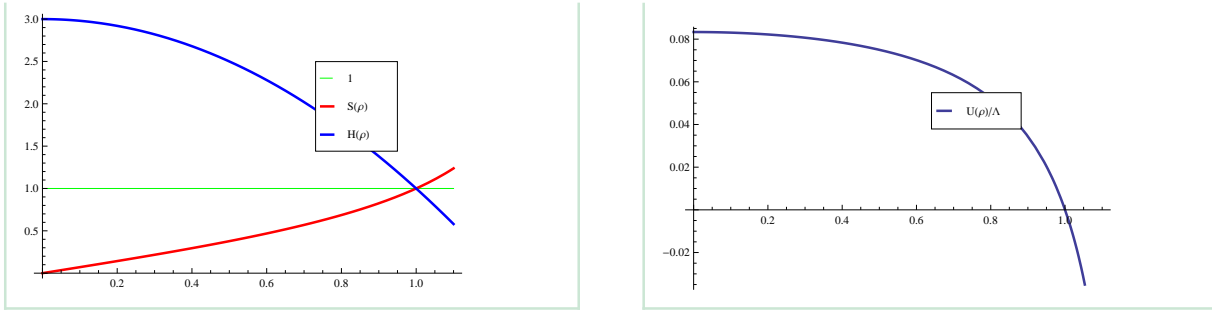


Figure 3: The CPW solution.

We have analysed the solution space with $U(\rho_0) = 0$ by varying h_1 . For $h_1 < -4$, the solutions are similar to the one in Fig. 4 in particular they have a diverging warp factor $H(\rho)$.

On the other side, solutions for $h_1 > -4$ look qualitatively as in Fig. 5 and have diverging $U(\rho)$. It is clear that the CPW solution is at the interface between the two types of solutions. In particular, $U(\rho)$ in the CPW solution has finite value at $\rho \rightarrow 0$. Also note that in this singular solution, S and H have the asymptotics of the form (46), while the regular CPW solution have asymptotics (47).

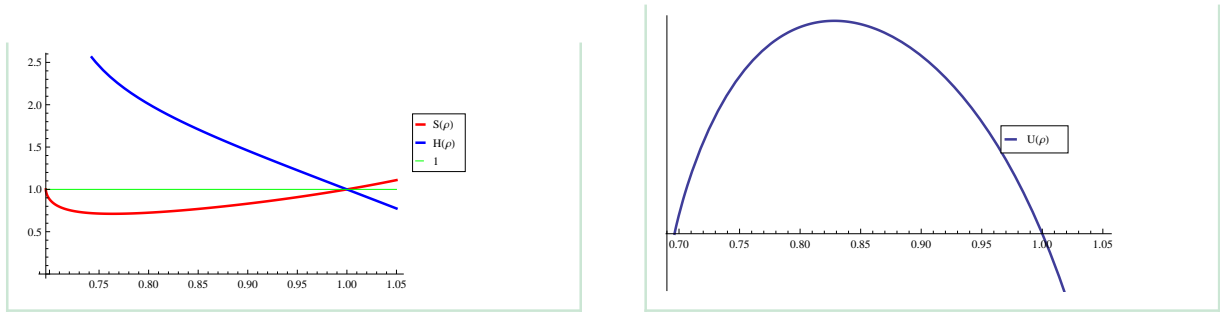


Figure 4: Solutions for $\rho_0 = 1$ and $h_1 = -4.5$. The left zero is around $\rho = 0.696287$.

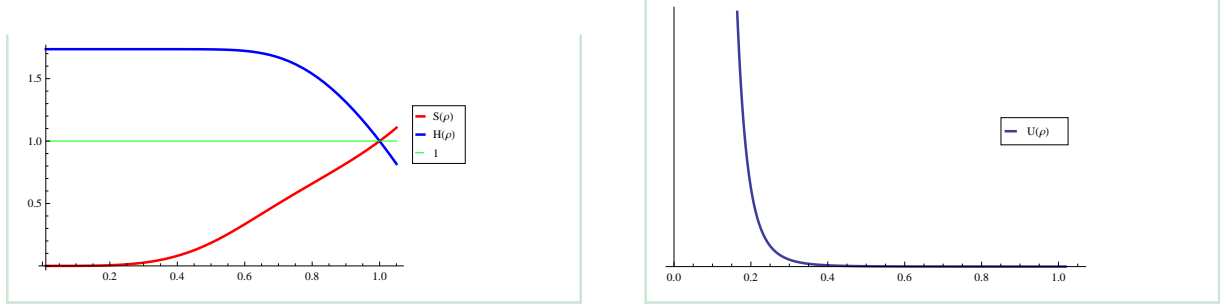


Figure 5: Solutions for $\rho_0 = 1$ and $h_1 = -3.5$. The singular point is at $\rho = 0$.

4.4 Solutions with $U(\rho_1) = 0$

Here we find one new numerical solution for $\rho \in I = [0, \rho_1]$. This is a completely new solution and is a central result of this paper. We obtain the solution “shooting” from $\rho = 0$ with initial velocities determined from (47) by a choice of the parameter s . Empirically, we find that the numerical solutions seems to be singular unless s is restricted to $1/\sqrt{2} < s < \sqrt{3/2}$ and within this range we find solutions like the ones in Fig. 6. We know that the target of this shooting algorithm must be $H(\rho_1) = S(\rho_1) = 1$ at $\rho_1 = \frac{\sqrt{3}}{2}$ and this requires delicate adjustment of the initial data. We were not able to resolve precisely the value of s that results in exactly this target data but it is evident from Fig. 6 that $s \approx 0.8247$.

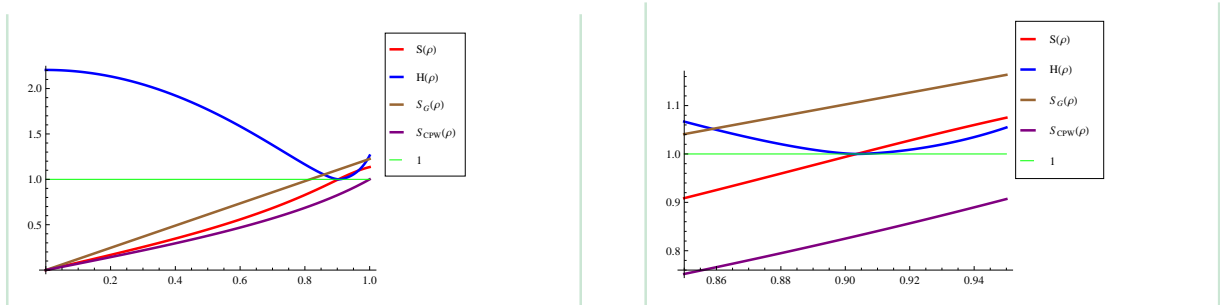


Figure 6: Shooting from $\rho = 0$ with $s = 0.8247$.

This solution has the topology of S^7 and is a natural generalization of the CPW solution. It would be interesting to demonstrate that this solution is the IR limit of a holographic RG flow.

4.5 Solutions with $U(\rho_2) = 0$

We also find a solution space at $\rho_2 = \sqrt{2/3}$ ($h_1 = h_2 = 0$), the solutions are parametrized by $h_3 = h$. The lowest terms in the series expansion are:

$$\begin{aligned} s_1 &= \sqrt{\frac{3}{2}}, \quad s_2 = s_3 = 0, \quad s_4 = -\frac{3}{4}\sqrt{\frac{3}{2}}h, \quad s_5 = -\frac{9}{40}h, \quad s_6 = -\frac{21}{16}\sqrt{\frac{3}{2}}h, \\ h_1 &= h_2 = 0, \quad h_3 = h, \quad h_4 = \frac{3}{2}\sqrt{\frac{3}{2}}h, \quad h_5 = \frac{9}{8}h, \quad h_6 = \frac{1}{32}(3\sqrt{6}h + 40h^2). \end{aligned} \quad (50)$$

One solution in this class is the analytic solution (28) for $c = 0$:

$$S(\rho) = \sqrt{\frac{3}{2}}\rho, \quad H(\rho) = 1. \quad (51)$$

which is plotted in Figs. 1 and 2. This also belongs to the second family around $\rho = 0$ with $s = \sqrt{3/2}$ while around the rightmost endpoint it has $h_k = 0$. This solution is, of course, nothing other than the round S^7 .

We have scanned the solution space in this class and have found evidence that once more we have two generic types of singular solutions with the analytic solution lying in between. Representatives of these two classes are plotted in Fig. 7 and 8

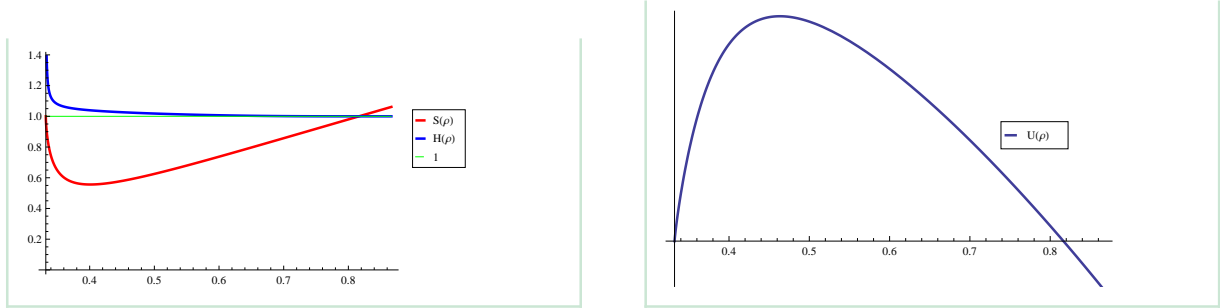


Figure 7: Solutions for $\rho_2 = \sqrt{2/3}$ and $h = -1$. The singular point is at $\rho = 0.332142$.

4.6 Solutions with $U(\rho_k) = 0$, $k \geq 3$

For ρ_k , $k \geq 3$ solutions are parametrized by $h_{k+1} = h$, with $h_1 = \dots = h_k = 0$. For each k , there is an analytic solution in the family (28) with $h_i = 0$, $\forall i$ (i.e. $H(\rho) = 1$). These are, of course, precisely the GMSW metrics and an example is plotted in Fig. 9 for $k = 3$.

For solutions with fluxes we must have $H(\rho) \neq 1$ and the interval I must continue all the way to $\rho = 0$. We have been unable to produce a numerical solution that map onto (47) at

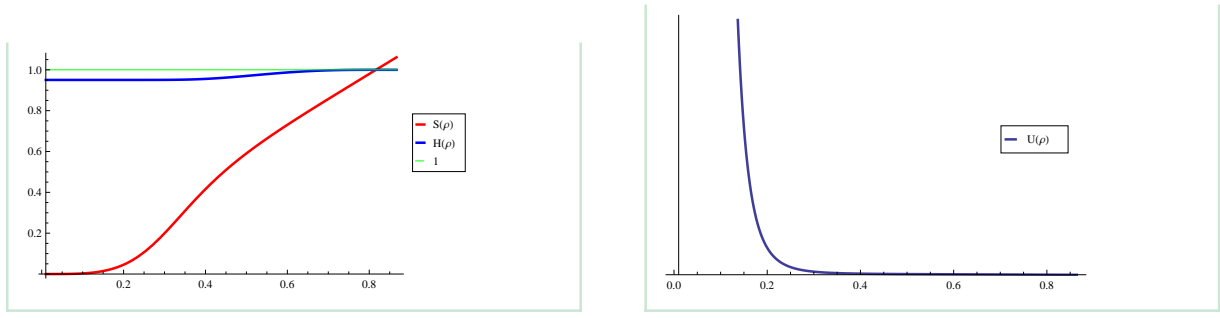


Figure 8: Solutions for $\rho_2 = \sqrt{2/3}$ and $h = 2$. The singular point is at $\rho = 0$.

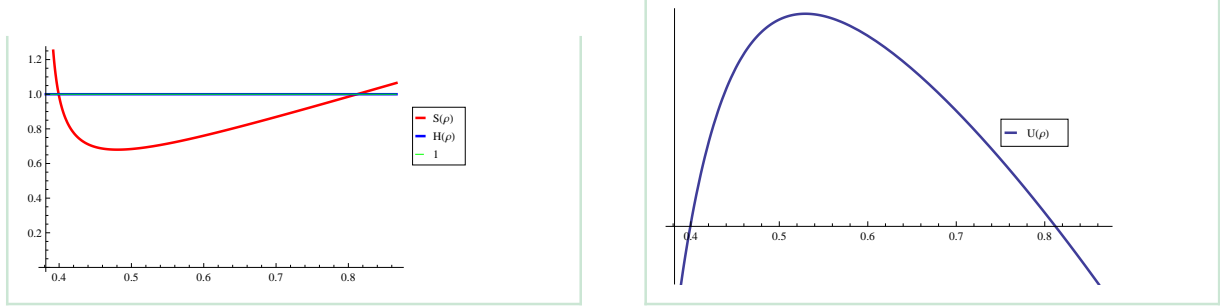


Figure 9: Solutions for $\rho_3 = \sqrt{5/2}/2$ and $h = 0$. Both ends are analytic.

$\rho = 0$ but we have been able to find solutions that map onto (46) however, as noted above, the latter have badly singular metrics because $U(\rho)$ diverges as $\rho = 0$.

4.7 Summary

After examining all the numerical solutions we find precisely one new, smooth solution: It is the one exhibited in Section 4.4 and has non-trivial flux and has $\rho \in I = [0, \rho_1]$. All other solutions appear to be singular.

5 Discussion

We have found a new $\mathcal{N} = 2$ supersymmetric, AdS_4 flux solution in M theory in which the topology of the compactification manifold is still S^7 . It appears as a single, isolated regular solution in a broad class that generalizes the CPW solution [21]. Given the form of our new solution and the close parallels between the CPW solution and the $\mathcal{N} = 2$ supersymmetric, AdS_5 flux solution in IIB supergravity [34, 35] (the PW solution) it is natural to ask whether there is another solution to IIB supergravity that is analogous to the solution found here. Indeed, there is a similarity between the equations (27) we have studied here and the general study of AdS_5 solutions of IIB supergravity in [36].

In fact, following [37], we have analyzed a IIB version of the Ansatz

$$ds_{10}^2 = H^{1/2} ds_{AdS_5}^2 - L^2 H^{-1/2} ds_5^2, \quad (52)$$

$$ds_5^2 = X_2^2 d\rho^2 + \rho^2 ds_{\mathbb{CP}^1}^2 + (X_4 \eta_{\mathbb{CP}^1} + X_5 d\phi)^2 + X_6^2 d\phi^2, \quad (53)$$

with a constant dilaton, a “holomorphic” $G^{(3)}$ flux as in [37], and

$$F^{(5)} = dC^{(4)} + *dC^{(4)}, \quad C^{(4)} = \frac{1}{4} H^4 \cos \beta dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3, \quad (54)$$

where β is the dielectric polarization angle. The resulting BPS equations are¹

$$\begin{aligned} \frac{dS}{d\rho} &= (n+2) \frac{S}{\rho} - (n+2) \frac{\lambda}{\Lambda} H \frac{S^3}{\rho^3}, \\ \frac{dH}{d\rho} &= (n+2) \frac{\lambda}{\Lambda} (1-H^2) \frac{S^4}{\rho^3(1-S^2)}, \end{aligned} \quad (55)$$

with $n = 1$ and $\lambda/\Lambda = 2/3$. The same equations with $n = 2$ and $\lambda/\Lambda = 1/2$ are the ones studied in this paper (27). The metric functions are given by

$$X_2 = -\sqrt{H} \frac{S}{\rho \sqrt{1-S^2}}, \quad X_4 = \frac{1}{\sqrt{H}} \frac{\rho^2}{S}, \quad X_5 = \frac{2}{3} \frac{1}{\sqrt{H}} S, \quad X_6 = \frac{2}{3} \sqrt{H} \sqrt{1-S^2}. \quad (56)$$

Interestingly the zeros of $g_{\rho\rho}^{-1}$ are still given by (39)

$$\rho_j = \sqrt{\frac{j+2}{2j+2}}. \quad (57)$$

The round S^5 solution is

$$S(\rho) = \rho, \quad H(\rho) = 1, \quad (58)$$

and solution of [34] corresponds to

$$S(\rho) = \frac{\rho}{\sqrt{2-\rho^2/2}}, \quad H(\rho) = 2 - \frac{3}{4}\rho^2. \quad (59)$$

One interesting conclusion of our study is that the solution of section 4.4 has no analogue in IIB. For the PW solution and the round S^5 solution in IIB, the interval I ends on the right at ρ_0 and ρ_1 respectively, there is no intermediate point whereas in the M-theory solutions of this paper, the CPW point and the round point end at ρ_0 and ρ_2 , leaving a new point ρ_1 corresponding the new solution presented here.

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¹We have checked that (55) agrees with eqs (5.5) of [36] after appropriate redefinitions.

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